SOME IDENTITIES OF SYMMETRY FOR q-BERNOULLI POLYNOMIALS UNDER SYMMETRIC GROUP OF DEGREE n

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Abstract. In this paper, we give some new identities of symmetry for q-Bernoulli polynomials under the symmetric group of degree n arising from p-adic q-integrals on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [7]}). \tag{1.1}$$

From (1.1), we have

$$qI_q(f_1) - I_q(f) = (q-1) f(0) + \frac{q-1}{\log q} f'(0), \quad \text{where } f_1(x) = f(x+1). \tag{1.2}$$

As is well known, the Bernoulli numbers are defined by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing B^n by B_n (see [1–12]). The Bernoulli polynomials are given by

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (\text{see [12]}). \tag{1.3}$$

In [3], L. Carlitz considered the q-analogue of Bernoulli numbers as follows:

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

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He also defined q-Bernoulli polynomials as follows:

$$\beta_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]_q^{n-l}\beta_{l,q}\quad(\text{see [1–3, 9]}). \tag{1.5}$$

In [7], Kim proved the following integral representation related to Carlitz q-Bernoulli polynomials:

$$\beta_{n,q}(x)=\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_q(x),\quad(n\geq 0). \tag{1.6}$$

From (1.2), we note that

$$q\int_{\mathbb{Z}_p}[x+1]_q^n\,d\mu_q(x)-\int_{\mathbb{Z}_p}[x]_q^n\,d\mu_q(x)=\begin{cases}q-1 & \text{if } n=0\\ 1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}. \tag{1.7}$$

By (1.7), we get

$$\beta_{0,q}=1,\quad q\beta_{n,q}(1)-\beta_{n,q}=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}.$$

The purpose of this paper is to give identities of symmetry for Carlitz's q-Bernoulli polynomials under the symmetric group of degree n arising from p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identiites of $\beta_{n,q}(x)$ under S_n

For $n\in\mathbb{N}$, let $w_1,w_2,\ldots,w_n\in\mathbb{N}$. Then, we have

(2.1)

$$\begin{aligned}&\int_{\mathbb{Z}_p}e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]_q t}\,d\mu_{q^{w_1w_2\cdots w_{n-1}}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[p^N]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{y=0}^{p^N-1}e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]_q t}\\
&\quad\times q^{\left(\prod_{j=1}^{n-1}w_j\right)y}.\\
&=\lim_{N\to\infty}\frac{1}{[w_np^N]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}\\
&\quad\times e^{\left[\left(\prod_{j=1}^{n-1}w_j\right)(m+w_ny)+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j\right]_q t}q^{w_1w_2\cdots w_{n-1}(m+w_ny)}.\end{aligned}$$

Thus, by 2.1, we get

(2.2)

$$\frac{1}{\left[\prod_{l=1}^{n-1}w_l\right]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\prod_{j=1}^{n} w_j x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{q^{w_1 w_2\cdots w_{n-1}}}(y)$$

$$= \lim_{N\to\infty} \frac{1}{\left[\prod_{l=1}^{n} w_l p^N\right]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{m=0}^{w_n-1} \sum_{y=0}^{p^N-1} q^{\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j w_n}$$

$$\times e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)(m+w_n y)+\left(\prod_{j=1}^{n} w_j\right)x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t}.$$

We note that (2.2) is invariant under any permutation $\sigma \in S_n$. Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, \ldots, w_n \in \mathbb{N}$, the following expressions*

$$\frac{1}{\left[\prod_{l=1}^{n-1} w_{\sigma(l)}\right]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q^{w_{\sigma(n)} \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}$$

$$\times \int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_{\sigma(j)}\right)y+\prod_{j=1}^{n} w_j x+w_{\sigma(n)} \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j\right]_q t} d\mu_{q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}(y)$$

are the same for any $\sigma \in S_n$.

We observe that

$$\begin{aligned}
&\left[\left(\prod_{j=1}^{n-1} w_j\right) y + \left(\prod_{j=1}^{n} w_j\right) x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q \\
&= \left[\prod_{j=1}^{n-1} w_j\right]_q \left[y + w_n x + \frac{w_n}{w_1}k_1 + \cdots + \frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1\cdots w_{n-1}}} \\
&= \left[\sum_{j=1}^{n-1} w_j\right]_q \left[y + w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j}k_j\right]_{q^{w_1\cdots w_{n-1}}}.
\end{aligned} \tag{2.3}$$

Thus, by (2.3), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{\left[\left(\prod_{j=1}^{n-1} w_j\right)y+\left(\prod_{j=1}^{n} w_j\right)x+w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q t} d\mu_{q^{w_1\cdots w_{n-1}}}(y) \\
&= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\right]_{q^{w_1\cdots w_{n-1}}}^m d\mu_{q^{w_1\cdots w_{n-1}}}(y) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left[\prod_{j=1}^{n-1} w_j\right]_q^m \beta_{m,q^{w_1\cdots w_{n-1}}}\left(w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j}k_j\right) \frac{t^m}{m!}.
\end{aligned} \tag{2.4}$$

For $m \geq 0$, from (2.4), we have

$$\int_{\mathbb{Z}_p} \left[\left(\prod_{j=1}^{n-1} w_j \right) y + \left(\prod_{j=1}^{n} w_j \right) x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q^m d\mu_{q^{w_1 \cdots w_{n-1}}}(y) \tag{2.5}$$

$$= \left[\prod_{j=1}^{n-1} w_j \right]_q^m \beta_{m, q^{w_1 \cdots w_{n-1}}} \left(w_n x + \sum_{j=1}^{n-1} \frac{w_n}{w_j} k_j \right).$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *For $m \geq 0$, $w_1, \dots, w_n \in \mathbb{N}$, the following expressions*

$$\left[\prod_{j=1}^{n-1} w_{\sigma(j)} \right]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q^{\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_{\sigma(i)} \right) k_j w_{\sigma(n)}}$$

$$\times \beta_{m, q^{w_{\sigma(1)} \cdots w_{\sigma(n-1)}}} \left(w_{\sigma(n)} x + w_{\sigma(n)} \sum_{j=1}^{n-1} \frac{k_j}{w_{\sigma(j)}} \right)$$

are the same for any $\sigma \in S_n$.

It is easy to show that

$$\left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_{q^{w_1 \cdots w_{n-1}}} \tag{2.6}$$

$$= \frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j \right]_q} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}$$

$$+ q^{w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} [y + w_n x]_{q^{w_1 \cdots w_{n-1}}}.$$

From (2.6), we can derive the following equation:

(2.7)

$$\int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_{q^{w_1 \cdots w_{n-1}}}^m d\mu_{q^{w_1 \cdots w_{n-1}}}(y)$$

$$= \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j \right]_q} \right)^{m-l} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{m-l} q^{l w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j}$$

$$\times \int_{\mathbb{Z}_p} [y + w_n x]_{q^{w_1 \cdots w_{n-1}}}^l d\mu_{q^{w_1 \cdots w_{n-1}}}(y)$$

(2.8)

$$
\begin{aligned}
&= \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[w_n]_q}{\left[\prod_{j=1}^{n-1} w_j\right]_q} \right)^{m-l} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{m-l} \\
&\quad \times q^{l w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j} \beta_{l,q^{w_1\cdots w_{n-1}}} (w_n x) .
\end{aligned}
$$

Thus, by (2.7), we get

$$
\begin{aligned}
&\left[\prod_{j=1}^{n-1} w_j \right]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j} \\
&\times \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_{q^{w_1\cdots w_{n-1}}}^{m} d\mu_{q^{w_1\cdots w_{n-1}}} (y) \\
&= \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j \right]_q^{l-1} [w_n]_q^{m-l} \beta_{l,q^{w_1\cdots w_{n-1}}} (w_n x) \\
&\quad \times \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_s-1} q^{\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j w_n (l+1)} \left[\prod_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{m-l} \\
&= \sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_j \right]_q^{l-1} [w_n]_q^{m-l} \beta_{l,q^{w_1\cdots w_{n-1}}} (w_n x) \, T_{m,q^{w_n}} (w_1, w_2, \ldots, w_{n-1} \mid l) ,
\end{aligned}
$$

where

$$
\begin{aligned}
&T_{m,q} (w_1, w_2, \ldots, w_{n-1} \mid l) \\
&= \prod_{s=1}^{n-1} \sum_{k_s=0}^{w_s-1} q^{(l+1) \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i\neq j}}^{n-1} w_i \right) k_j \right]_q^{m-l} .
\end{aligned}
$$

As this expression is invariant under any permutation in S_n, we have the following theorem.

Theorem 2.3. *For $m \geq 0$, $n, w_1, \ldots, w_n \in \mathbb{N}$, the following expressions*

$$
\begin{aligned}
&\sum_{l=0}^{m} \binom{m}{l} \left[\prod_{j=1}^{n-1} w_{\sigma(j)} \right]_q^{l-1} \left[w_{\sigma(n)} \right]_q^{m-l} \beta_{l,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}} \left(w_{\sigma(n)} x \right) \\
&\times T_{m,q^{w_{\sigma(n)}}} \left(w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} \mid l \right)
\end{aligned}
$$

are all the same for $\sigma \in S_n$.

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